Combinatorics ⇒ science of counting things

Sum Rule: if we have two groups of objects, and the two groups don't share any, # in both = # 1st + #2nd

Ex: 3 choices of sports, 2 choices of video game
$3+2=\boxed{5}$ choices of entertainment

Product Rule:

If you have k choices to make, and there are n_i options for the i^{th} choice, then the total # of choices is $n_1 \cdot n_2 \cdot \ldots \cdot n_k$

Ex: 3 shirts, 4 shorts. How many possible outfits? $3 \cdot 4 = \boxed{12}$.

Ex: # of different 5 digit numbers (including ones that start w/ 0s.

$\frac{\quad}{10}$ $\frac{\quad}{10}$ $\frac{\quad}{10}$ $\frac{\quad}{10}$ $\frac{\quad}{10}$

5 positions, 10 choices of digits for each.

$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = \boxed{10^5}$

Ex: # of different 5 digit numbers that have no repeat digits.

$\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$

10 9 8 7 6

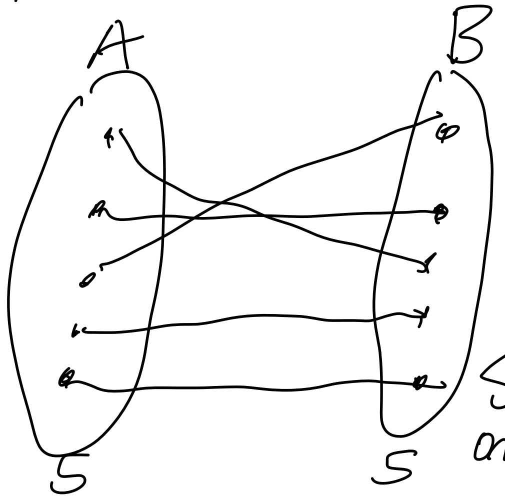
We have 10 choices to start, but since we can't repeat digits, we have one less for each successive choice.

$\boxed{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}$ or since

$n! = n \cdot (n-1) \cdots 2 \cdot 1$

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \boxed{\frac{10!}{5!}}$$

Bijection Rule:



Creating a one-to-one "correspondence" or "translation" between sets A and B shows

size of A = size of B

one-to-one correspondence = bijection

Ex: # of possible subsets of a set of n items.

Full set: $\{1, 2, 3, \ldots n\}$

Ex. of subsets: empty set and full set count as subsets

$\{1, 5, 7, n-4\}$ $\{\}$ $\{1, 2, 3, \ldots, n\}$

One-to-one between subsets and list of 0s and 1s of length n. Include 0 in list for excluded elements and 1 in list for included elements in subset.

Example: $\{1, 3, n\}$

$\{1, 2, 3, 4, \ldots, n\}$

$\{1, 0, 1, 0, \ldots, 1\}$

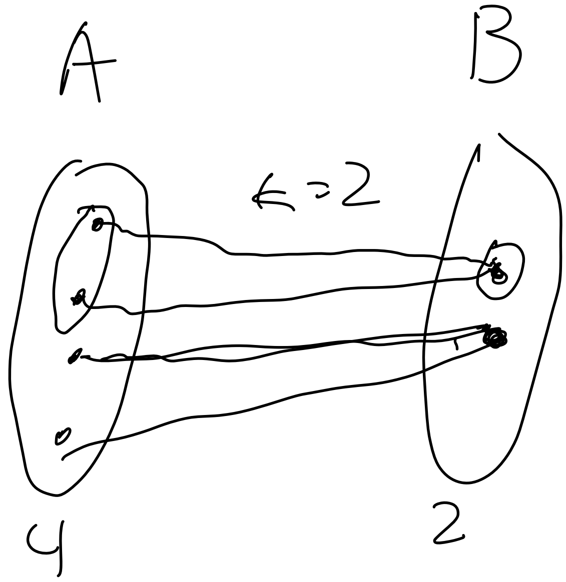
Count number of lists of 0s and 1s of size n

$\underline{\quad}\ \ \underline{\quad}\ \ \underline{\quad}\ \ \underline{\quad}\ \ \ldots\ \ \underline{\quad}$

$2 \cdot 2 \cdot 2 \cdot 2 \cdot \quad \cdot 2$

2 choices for each position. n positions. Product rule gives: $\boxed{2^n}$

Division Rule: If you create a k to 1 correspondence from A to B, then



$$\text{size of } B = \frac{\text{size of } A}{k}$$

$$2 = \frac{4}{2}$$

Ex: Formula for $_nC_k$ or $\binom{n}{k}$ ⇒ # of ways you can choose k items from a set of n items, when order doesn't matter

Step 1: Count # of ways you can choose an ordered list of k items from n total.

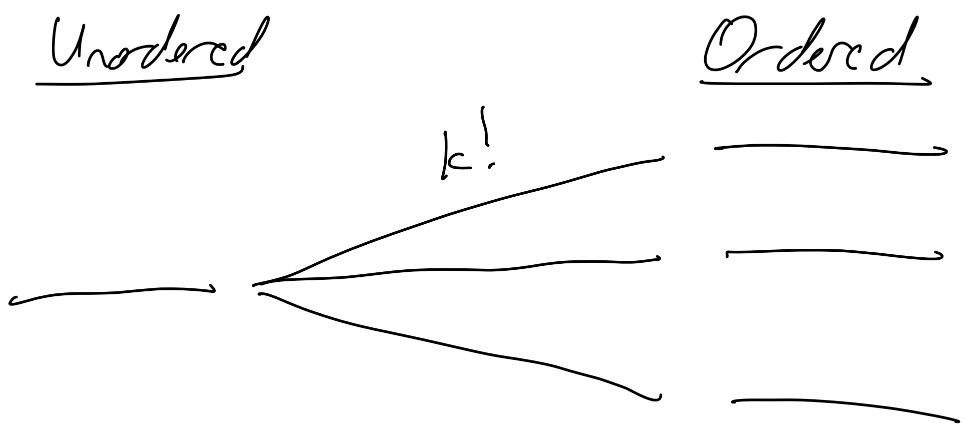
$$\underline{\{1, 2, 3, \ldots n\}}$$

$$\underset{1^{st}}{\underline{n}} \cdot \underset{2^{nd}}{\underline{n-1}} \cdot \underset{3^{rd}}{\underline{n-2}} \cdot \underset{4^{th}}{\underline{n-3}} \cdots \underset{k^{th}}{\underline{n-(k-1)}}$$

$$= \frac{n(n-1)(n-2)\cdots(n-(k-1))\cdot n-k \cdots 1}{(n-k)\cdots 1}$$

$$= \frac{n!}{(n-k)!}$$

Step 2: Create correspondence between set of ordered lists and set of unordered groups of k items.



For every item in unordered list, we can reorder it in $k!$ ways. So $k!$ to 1 correspondence. By division rule,

$$\# \text{unordered} = \frac{\# \text{ordered}}{k!} = \frac{n!}{(n-k)!\,k!}$$

$$\boxed{{}_nC_k = \frac{n!}{(n-k)!\,k!}}$$

Proofs w/ Combinatorics.

1) Pigeonhole Principle

If we have n holes for pigeons and a pigeons and $a > n$, at least 2 pigeons share a hole.

Ex: At most 120,000 hairs on human

6.873 million in MA

Prove that at least 2 people in MA have the same number of hairs on head.

<u>Proof.</u> Use pigeonhole principle.

<u>Pigeons:</u> people in massachusetts = 6.835 mil

<u>Pigeonholes:</u> different numbers of hairs that can be on a person's head. Ranges from 0 to 120k, so 120k different possible numbers

By pigeonhole principle, 2 people share same # of hairs on head. $\boxtimes$

Combinatoric Proof

Say you have a collection of objects.

1) Count set with method 1: says there are x items.
2) Count set with method 2: says there are y items.

Then $x = y$, since some set can't have two different counts.

Ex: Show that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

Proof. Use combinatorial proof.

Count the number of different subsets of a set of n items.

Method 1: We previously counted this, and we got $\boxed{2^n}$.

Method 2:

Number of different subsets that have k items: $\binom{n}{k}$

Sets will be disjoint since one subset can't have two different sets of items. Sum across all possible k to get count.

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\dots\binom{n}{n}$$

$$\boxed{\sum_{i=0}^{n}\binom{n}{i}}$$

Therefore, $\displaystyle\sum_{i=0}^{n}\binom{n}{i}=2^n$ $\boxtimes$